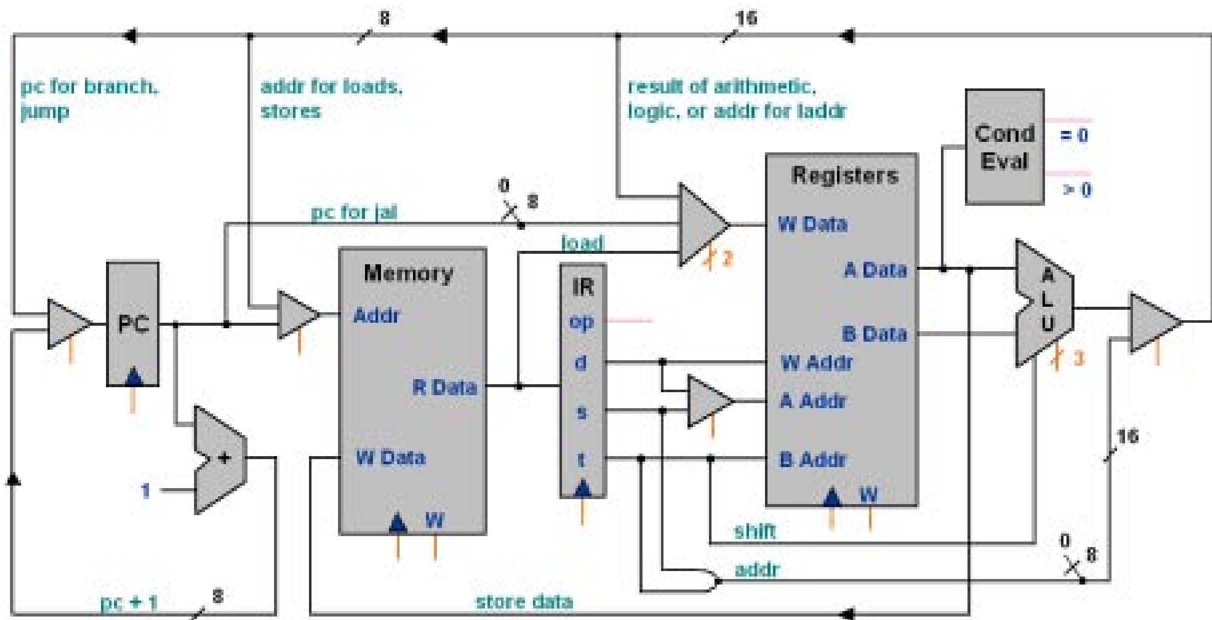
Circuits*

October 16, 2003

Abstract

In this document, we build the TOY machine in hardware. Along the way we introduce Boolean functions, combinational circuits, and sequential circuits. Circuits are used to model the hardware of digital computers. They correspond directly with devices that we can build. In this document, we address the following questions:

- How can we build logical gates from wires and transistors?
- How can we build arithmetic circuits from logical gates?
- How can we build memory circuits?
- How can we combine all of these components to build a computer?



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1 Boolean Logic

Boolean logic is the foundation of digital computers. It is a rigorous mathematical system based on the two distinct values *true* and *false*. Boolean logic is useful whenever there are two possible situations. A light bulb is either on or off. In mathematics, a statement is either true or false. In digital circuits, a capacitor is either charged or uncharged. We universally associate 1 with `on` or `true`, and 0 with `off` or `false`, and refer to the two binary digits as *bits*.

A *mathematical function* is an object that takes an input and produces an output. For a given function f, the same input always produces the same output. Some familiar functions that transform one real input into one real output include: $f(x) = x^2 + 5$, $f(\theta) = \sin\theta$, $f(x) = 9x/5 + 32$. Functions can also take several inputs, e.g., $f(x, y) = x + y$, $f(x, y) = \sqrt{x^2 + y^2}$, $f(r, \theta) = r \sin\theta$. *Boolean functions* are functions whose inputs and output consists of only zeros and ones. For example, the NOT function takes one input bit and returns the complementary bit.

$$\text{NOT}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1. \end{cases}$$

We will consider several fundamental Boolean functions, including AND, OR, and XOR. Then we'll build more complicated functions from these primitive functions.

1.1 Boolean Functions of Two Variables

A Boolean function of two variables has two inputs, each of which can be 0 or 1. Thus, there are 4 possible input combinations. For each input combination, a Boolean function must specify either 0 or 1 as output. Thus, there are $2^4 = 16$ distinct Boolean functions of two variables. The most important ones are AND and OR.

The *AND function* takes two input bits, and outputs 1 if both inputs are 1; otherwise 0.

$$\text{AND}(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \\ 0 & \text{if } x = 0 \text{ and } y = 1 \\ 0 & \text{if } x = 1 \text{ and } y = 0 \\ 1 & \text{if } x = 1 \text{ and } y = 1. \end{cases}$$

For convenience, we often express a Boolean function more concisely using a *truth table.*

x	**y**	**AND**
0	0	0
0	1	0
1	0	0
1	1	1

Figure 1.1: The truth table for the AND function.

A truth table has a column for each of the inputs, and also for the Boolean function(s) that we are interested in describing. There is a row for each possible combination of input values. A function of two variables has 4 rows.

The *OR function* takes 2 input bits and outputs 1 if either input (or both) is 1; it outputs 0 if both inputs are 0. The *exclusive OR (XOR) function* is similar to the OR function, except that it returns 1 if *exactly* one of the inputs is 1 (but not both); otherwise it returns 0. Figure 1.2 gives a truth table for all 16 Boolean functions of two variables, specifying the name of the function whenever possible.

x	y	ZERO	AND		x		y	XOR	OR
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

x	y	NOR	EQ	y’		x’		NAND	ONE
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Figure 1.2: The truth table for all Boolean functions of two variables.

1.2 Boolean Functions of 3 or More Variables

Many applications involve functions of several variables. A Boolean function of three variables has three inputs, each of which can be 0 or 1. Thus, there are $2^3 = 8$ possible input combinations, and 8 rows in the its truth table. In general, a function of n variables has 2^n possible input combinations. For each input combination, a Boolean function must specify either 0 or 1 as output. Thus, there are $2^{2^3} = 256$ distinct Boolean functions of three variables. In general, there are 2^{2^n} different Boolean functions of n variables. It is not unusual in engineering applications for a function to have thousands or millions of inputs. Even for $n = 8$, it is prohibitively expensive to fabricate physical processes for each of the $2^{2^8} \geq 10^{77}$ such Boolean functions.

Generalizations of the AND and OR functions that take several inputs are important in designing circuits. An *n-way AND function* takes n inputs and outputs 1 if all n inputs are 1; otherwise it outputs 0. An *n-way OR function* takes n inputs and outputs 1 if at least one of the n inputs is 1; otherwise it outputs 0.

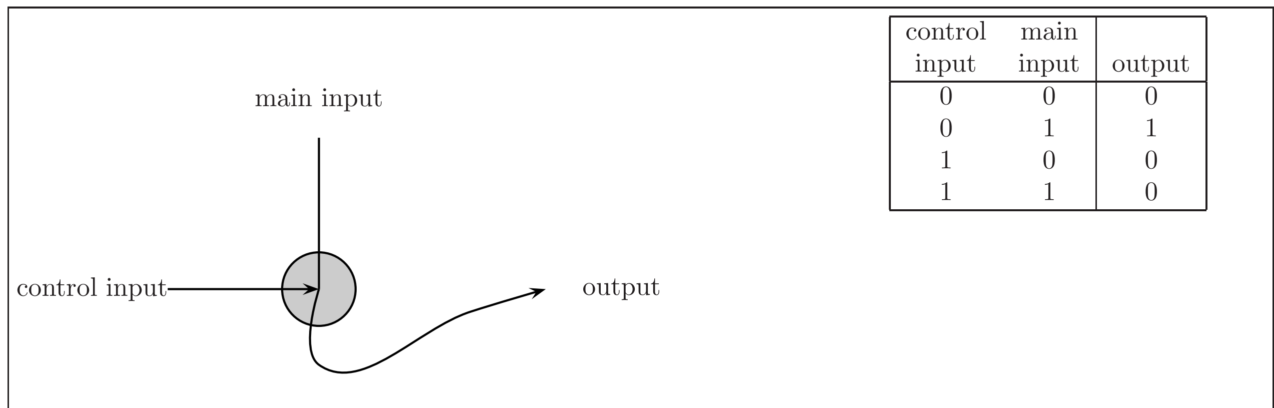
The following two Boolean functions play an important role in designing arithmetic circuits. The *n-way majority function (MAJ)* takes n input bits, and outputs 1 if at least half of the inputs are 1; otherwise it outputs 0. The *n-way odd parity function (ODD)* takes n inputs, and output 1 if an odd number of inputs are 1; otherwise it outputs 0. Figure 1.3 summarizes the 3-way versions of the AND, OR, MAJ, and ODD functions.

x	y	z	AND	OR	MAJ	ODD
0	0	0	0	0	0	0
0	0	1	0	1	0	1
0	1	0	0	1	0	1
0	1	1	0	1	1	0
1	0	0	0	1	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Figure 1.3: Some important Boolean functions of 3 variables.

2 Controlled Switches

Controlled switches are the fundamental building blocks of computer circuits, much like atoms are the fundamental building blocks of matter. A controlled switch has two inputs and one output, as illustrated in Figure 1. The *main input* is drawn on the top. The *control input* is drawn on the left; it controls whether or not the main input is able to pass through the controlled switch. If the control input is set to *pass* (0), then the output is the same as the main input; if it is *no pass* (1), then the output is 0 regardless of the main input.



control input	main input	output
0	0	0
0	1	1
1	0	0
1	1	0

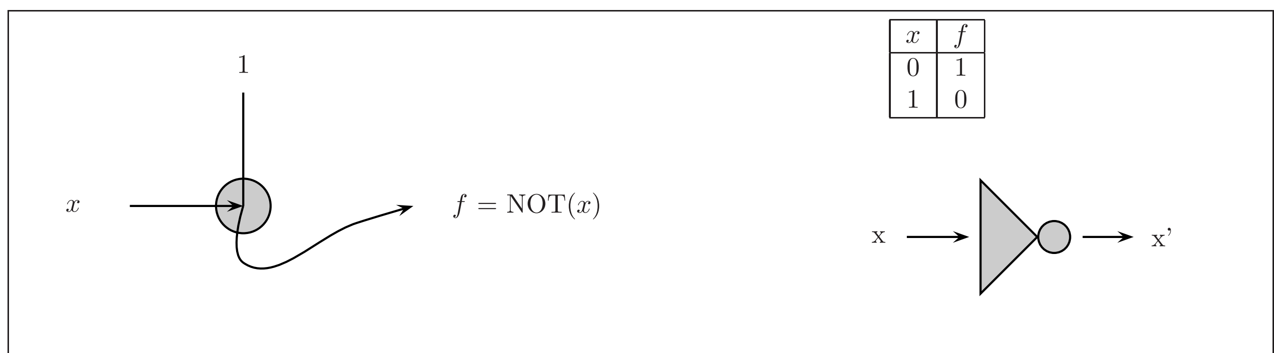
Figure 1: A controlled switch and corresponding truth table.

A controlled switch is an abstraction of a physical process. Modern digital circuits are built from wires and transistors, with current flowing through the circuit. However, circuits can also be built from other processes. For example, in a "hydraulic circuit" water flows through pipes. In such a hydraulic circuit, the control input of a switch is a valve; if it is closed then no water can pass through. Other types of controlled switches include vacuum tubes, relays, and DNA. The discovery of a faster physical switch would profoundly change the way computers are built; this is an active area of research. To gain a deeper understanding of computer circuits, it is convenient to abstract away the physical details of the hardware involved. For the rest of the document, we will assume that we have an idealized switch, and not worry about wiring, heat dissipation, electrical interference, circuit area, signal propagation delay, and other low level details.

3 Gates

A *gate* is a low-level circuit that is used as a primitive element in building more complicated circuits. Gates are built by combining controlled switches with wires. We describe how to build NOT, OR, and AND gates.

The NOT gate can be implemented using a single controlled switch whose main input is always on, as in Figure 2. The OR gate can be implemented using three controlled switches, as in Figure 3. The AND gate can be implemented using two controlled switches, as in Figure 4.[1]

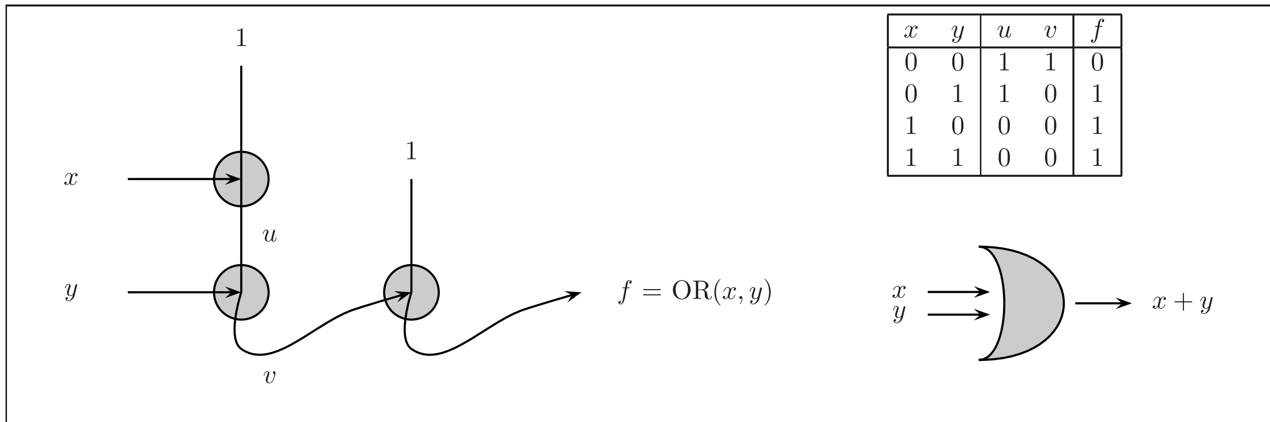


x	f
0	1
1	0

Figure 2: Implementing a NOT gate using one switch, its truth table, and its schematic symbol.

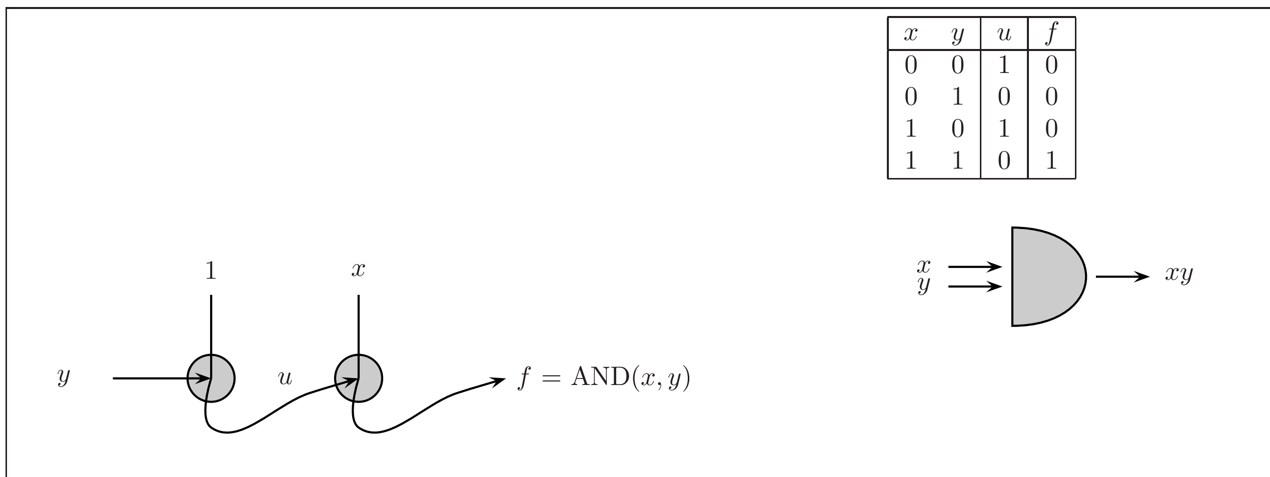
The AND, OR, and NOT gates are used so frequently that they have special schematic symbols. Instead of drawing the underlying controlled switches, it is standard to draw an AND gate as a half circle. We implicitly understand that the half circle represents 2 controlled switches, configured as in Figure 4. Similarly, we represent an OR gate with a crescent. A NOT gate is drawn as a triangle with a circle at its tip.

[1] Building an AND gates from pass transistors would require four transistors, since all exposed main inputs must be on.



x	y	u	v	f
0	0	1	1	0
0	1	1	0	1
1	0	0	0	1
1	1	0	0	1

Figure 3: Implementing an OR gate using three switches, its truth table, and schematic symbol. If x and y are both set to pass (0), then the control input of the third switch (v) is 1; otherwise it is 0. The third switch acts like a NOT gate, and its output is v'. This is exactly the OR function.



x	y	u	f
0	0	1	0
0	1	0	0
1	0	1	0
1	1	0	1

Figure 4: Implementing an AND gate using two switches, its truth table, and schematic symbol. The left switch acts like a NOT gate; it's output is y'. The second transistor has main input x and control input y'. It's output is 1 if and only if both x and y are both 1.

4 Combinational Circuits

Combinational circuits are physical hardware used to model Boolean functions. They are comprised of gates and wires in a *directed acyclic* configuration. Each wire (other than the input wires) connects the output of one gate to the input of another gate. There is no chain of wires and gates that form a directed cycle or loop. (See Figure 5.) This would cause the output of one gate to ultimately affect one of its inputs. Such a directed cycle is called a *feedback loop*. Circuits with feedback loops are called *sequential circuits*, and are considered in Section 6.

In a combinational circuit, each gate takes a prespecified number of input wires, evaluates the input according to the Boolean function associate with the gate, and transmits the value to the output wire. Here's a procedure for determining the output of a combinational circuit. Start with the input wires, whose values are known. Propagate these values along the wire to the gates to which they connect. Once all of the inputs to a gate are known, propagate the information along its output wire. Continue this process until the status of all wires and gates is known. Since combinational circuits have no feedback loops, this process is well-defined, and guarantees a unique value for each gate and wire upon termination.

In principle, it is possible to build a combinational circuit for any Boolean function directly from controlled switches. However, the resulting design could be overwhelmingly complex, due to the huge number of

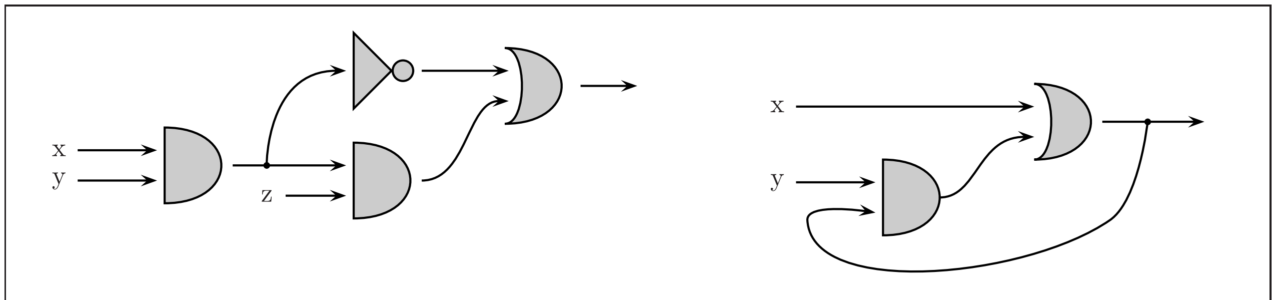


Figure 5: The left circuit is a combinational circuit since it has no feedback loops. The right circuit is a sequential circuit since it has a feedback loop.

switches. Instead, it is convenient to build circuits from somewhat higher level building blocks, namely the NOT, AND, and OR gates. This modular approach will facilitate the construction, as well as our understanding, of higher level circuits. By working at this higher level of abstraction, we can concentrate on designing more complicated circuits, without worrying about the lower level details. Of course, each of the NOT, AND, and OR gates that we use in the process will ultimately be physically realized with transistors (or other controlled switches). Again, depending on the type of circuit we are designing, we may wish to work at a still higher level of abstraction, and use higher level circuits as primitive circuit elements.

4.1 Multiway AND and OR gates

We describe how to build an n-way AND gate. Recall, an n-way AND gate outputs 1 if all of its inputs are 1; otherwise it outputs 0. First, we build a 3-way AND gate. We could start from switches, but instead, we build it by combining two 2-way AND gates. They key observation is that AND(x, y, z) = AND(x,AND(y, z)). Similarly, we can build an n-way AND gate using $(n - 1)$ 2-way AND gates. A 3-way and an 8-way AND gate are shown in Figure 6.

An n-way OR gate outputs 1 if at least 1 of its inputs is 1; otherwise it outputs 0. Designing an n-way OR gate is very similar to an n-way AND gate because OR(x, y, z) = OR(x,OR(y, z)).

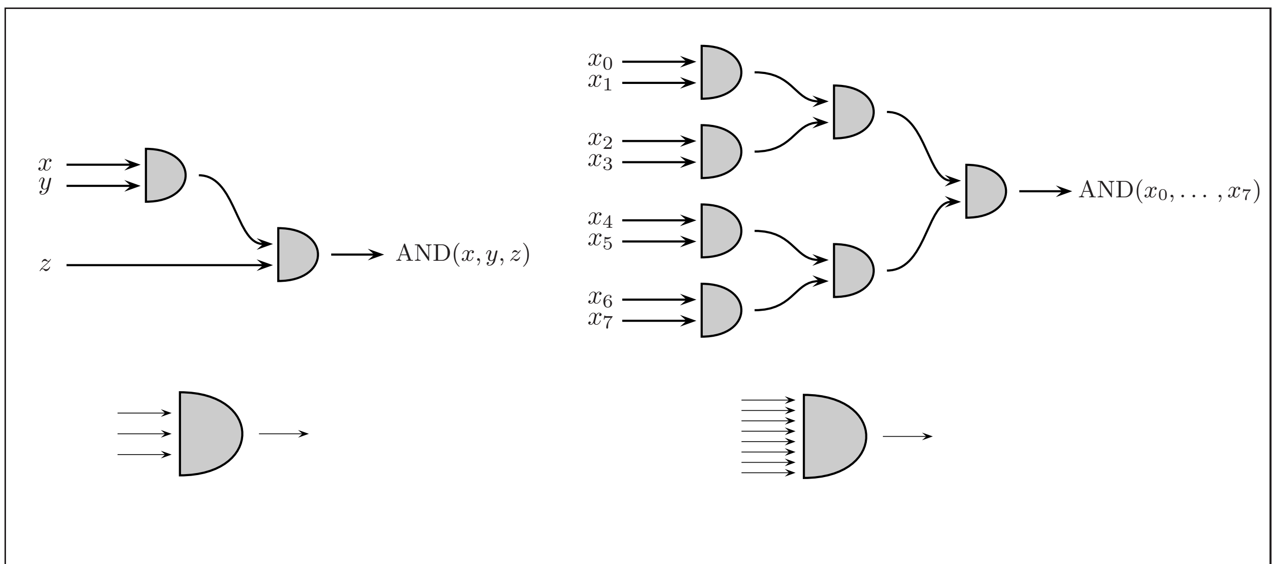


Figure 6: A 3-way AND gate built from two 2-way AND gates, an 8-way AND gate built from seven 2-way AND gates, and their schematic symbols.

4.2 Sum-of-Products Method

The *sum-of-products* method is a systematic process for designing any Boolean function using only AND, OR, and NOT gates. First we introduce some standard notation that will serve to explain the name "sum-of-products." We use the "product" xy to denote AND(x,y), the "sum" $x + y$ to denote OR(x,y), and x' to denote NOT(x). The product notation agrees with your intuition from multiplying integers, since the values are always 0 or 1. With the summation notation, $0 + 0 = 0$, $1 + 0 = 0 + 1 = 1$, as usual, but note that $1 + 1 = 1$. The interested reader can verify that many familiar arithmetic laws hold with this new definition of product and sum.[2] By convention, the NOT operation has the highest precedence, followed by AND, and then OR. So, for example, $xy' + z$ is correctly interpreted as $(x(y')) + z$.

We illustrate the sum-of-products method by designing an XOR gate using only AND, OR, and NOT gates. Recall the XOR(x,y) function is 1 if and only if exactly one of x or y is 1. We first consider two somewhat simpler functions. The function $x'y$ is 1 if and only if $x = 0$ and $y = 1$. The function xy' is 1 if and only if $x = 1$ and $y = 0$. Note that XOR(x,y) is 1 if and only if $x'y = 1$ or $xy' = 1$. Thus, the Boolean function $x'y + xy'$ is identical to the desired XOR(x,y) function.

x	y	XOR	$x'y$	xy'	$x'y + xy'$
0	0	0	0	0	0
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

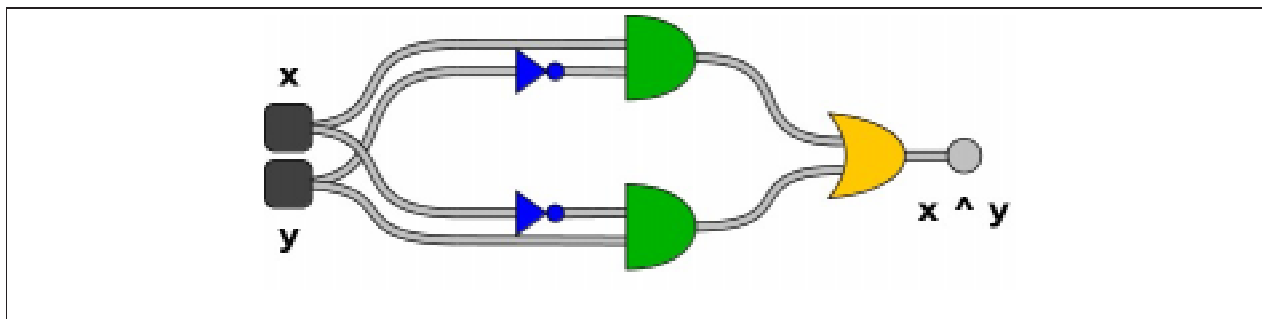


Figure 7: An XOR circuit.

The general sum-of-products procedure involves forming an AND term for each 1 in the original function's truth table column. The goal is to have each AND term be 1 if and only if the corresponding input row is 1. Finally all of these AND terms are OR'ed together, so that if any one of them is 1, then the OR expression is 1.

The sum-of-products method naturally extends to handle functions with more than 2 inputs. Recall the MAJ(x, y, z) function returns 1 if and only if 2 or more of its inputs are 1. Using the sum-of-products method, we form an AND term for each of the 4 input combinations for which MAJ(x, y, z) outputs 1: $(x, y, z) = (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)$. The four AND terms are $x'yz$, $xy'z$, xyz', and xyz, respectively. Thus MAJ$(x, y, z) = x'yz + xy'z + xyz' + xyz$.

x	y	z	MAJ	$x'yz$	$xy'z$	xyz'	xyz	$x'yz + xy'z + xyz' + xyz$
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

[2] For example, the usual commutative, distributative, and associative rules of algebra hold: $x + y = y + x$, $x \cdot y = y \cdot x$, $(x + y) + z = x + (y + z)$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x')' = x$. DeMorgan's identities $(ab)' = a' + b'$ and $(a + b)' = a'b'$ are also useful.

4.3 Better circuit design

It is possible to design different circuits that produce the same output, given the same input. The sum-of-products method provides a recipe for building combinational circuits. It is guaranteed to work, but may not always produce the “best” circuit. For example, a simpler majority circuit design is $\text{MAJ}(x, y, z) = xy + xz + yz$. See Figure 8.

There are two common quantitative criteria for measuring the quality of a circuit. The *size* of a circuit is the number of gates that it contains. This measure reflects the resources needed to build the hardware. The *depth* of a circuit is the length of the longest path from a circuit input to a circuit output. Physical circuits can propagate their values in parallel. The depth reflects the time needed to evaluate the circuit. Circuits with smaller depth run faster. The two circuits in Figure 8 both compute the majority function of three inputs. The first circuit has size 8 and depth 4, while the second has size 4 and depth 3. Designing circuits of small size and depth is an important and challenging problem, but we will not concentrate on these issues here.

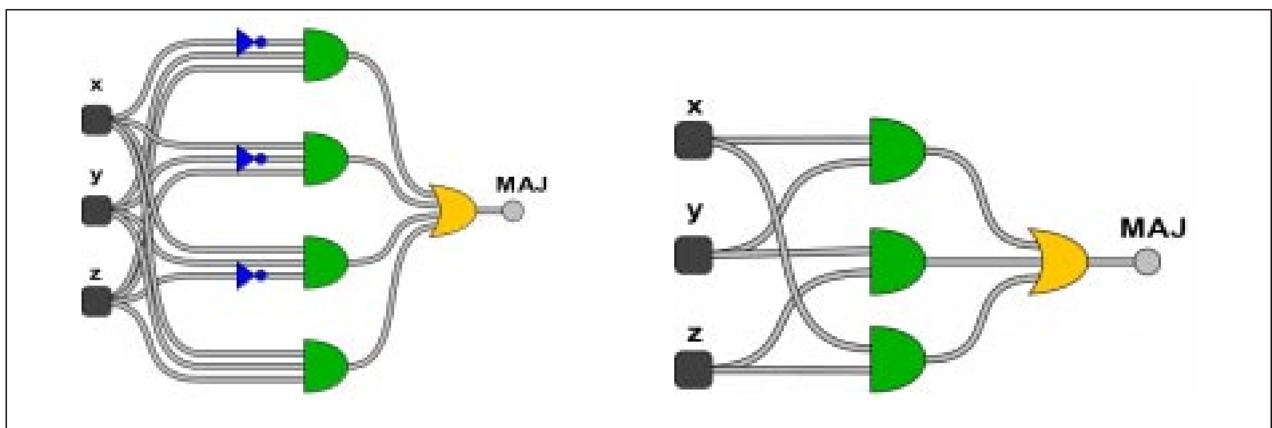


Figure 8: A 3-way majority circuit built using the sum-of-products method (left) and a simpler circuit (right).

4.4 Multiplexer

A multiplexer (or mux for short) is a useful circuit for selecting from one of several possibilities. An *n-bit multiplexer* has n select inputs, 2^n data inputs, and a single output. The n select inputs are interpreted as a binary number. The input wire (which are numbered from 0 to $2^n - 1$ from top to bottom) corresponding to this binary number is selected, and its value becomes the output of the multiplexer circuit. We will use multiplexers in section 6.8 to build TOY registers and main memory.

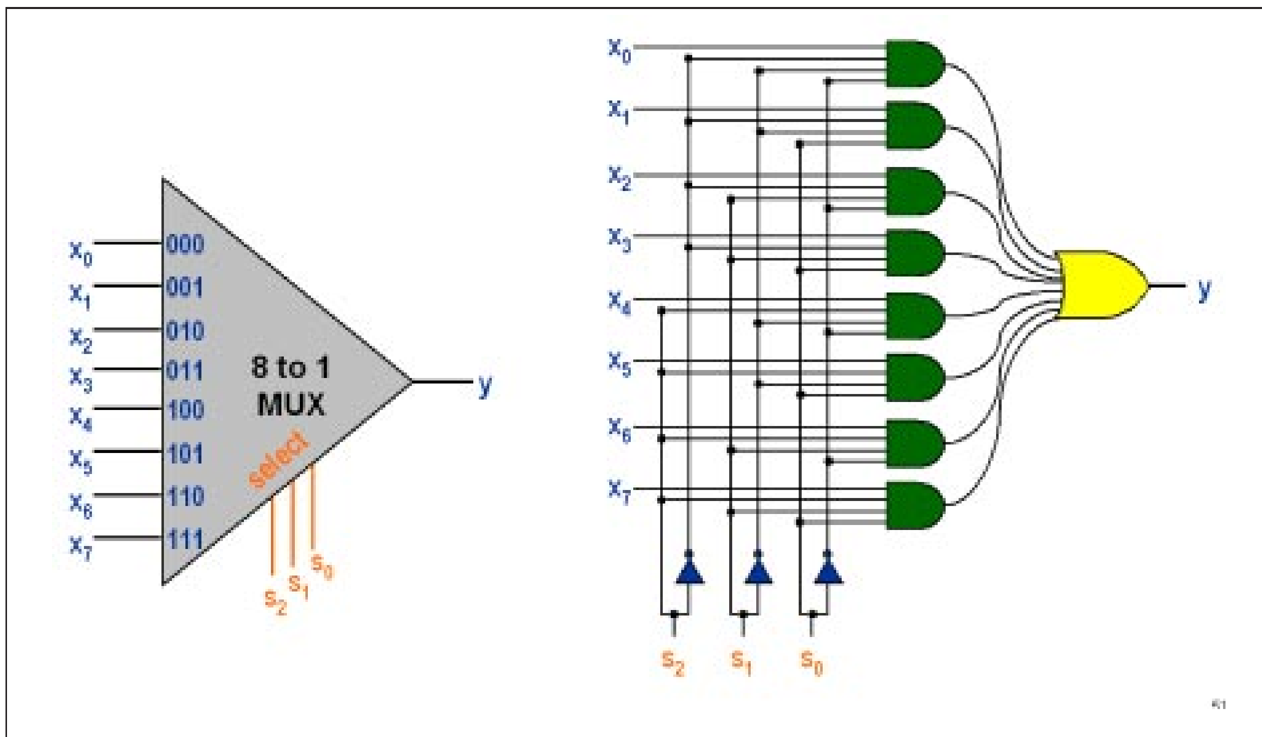


Figure 9: The interface and implementation of a 3-bit multiplexer.

Sometimes it is convenient to have a multiplexer select from a bank of n input buses, each of which contains k wires. Think of $n = 256$ as being the number of memory locations and $k = 16$ as being the number of bits stored in a register. This circuit is useful to extract the contents of one memory location. We call such a circuit a *k-wide n-to-1 multiplexer*. We can implement it by layering k n-to-1 multiplexers. Figure 10 shows a multiplexer that selects from one of two possible 4-bit buses.

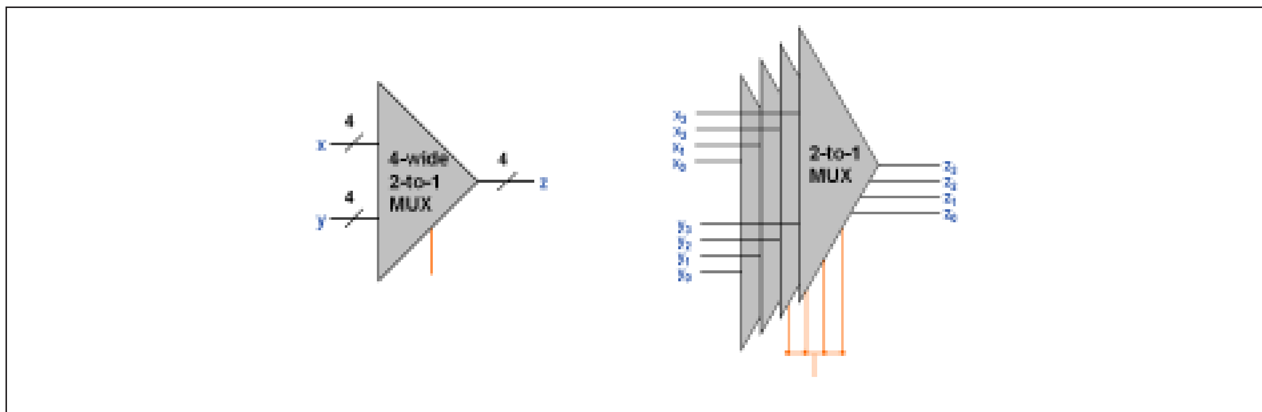


Figure 10: The interface and implementation of a 4-wide 2-to-1 multiplexer.

4.5 Decoder

A *decoder* is a circuit that converts from binary to unary. An *n-bit decoder* has n *select inputs* and 2^n outputs. See Figure 11 for a 3-bit decoder. Exactly one of the 2^n outputs is 1; the rest are 0. The select inputs determine which output is on. Interpret the n inputs as a binary number (read left to right). The output wire corresponding to this binary number is turned on; all other output wires are off. E.g., if the three inputs are 1, 1, and 0, then output wire number 6 is on. The outputs wires are numbered 0 to 2^n - 1 from top to bottom. In Section 6.8 we will use this circuit to build TOY registers and main memory.

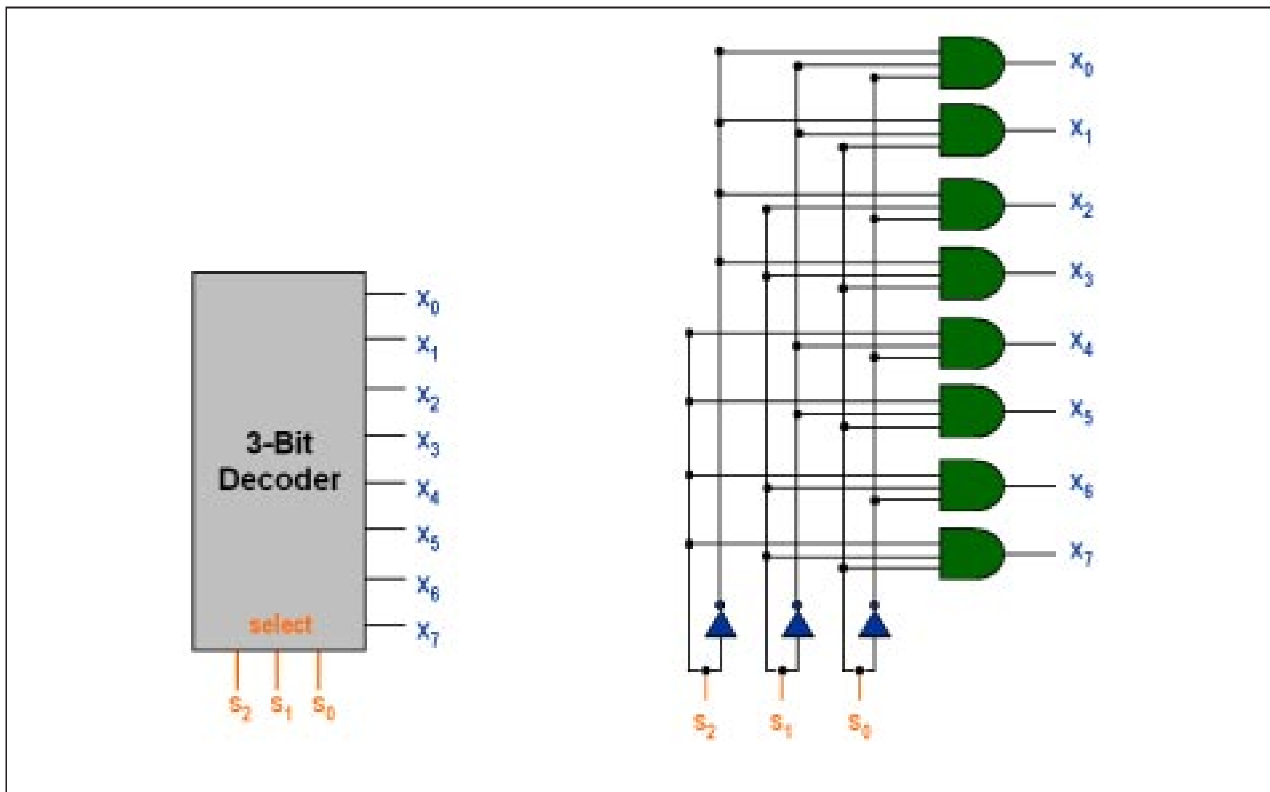


Figure 11: The interface and implementation of a 3-bit decoder.

4.6 Adder

We describe how to design a circuit to perform integer addition. This illustrates that simple Boolean circuits can be used to perform arithmetic.

Since computers store only zeros and ones, we build a circuit that adds in binary (base 2). Adding two numbers in binary is much like adding two numbers in decimal (base 10). In decimal, we first add the two 1's digits together; if the sum is 10 or more, we carry a 1 over to the 10's digit. Then we add the two 10's digits and the carry bit; if the resulting sum is 10 or more, we carry a 1 over to the $10^2 = 100$'s digit. This process is repeated until we are finished. In binary, everything is the same except that 10 is replaced with 2.

We need to formalize this procedure in order to build an adder circuit. Suppose we want to compute $s = x + y$, where x and y are nonnegative 8-bit binary integers. Note that in this section the symbol '+' denotes ordinary addition, not the Boolean OR function. We notate the ith digit of x, y and s by x_i, y_i, and s_i, respectively. E.g., if $x = 01100101$ then $x = x_7x_6x_5x_4x_3x_2x_1x_0$ and $x_0 = 1, x_1 = 0, x_2 = 1$, and so on. We wish to compute:

$$\begin{array}{ccccccccc} & x_7 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & x_0 \\ + & y_7 & y_6 & y_5 & y_4 & y_3 & y_2 & y_1 & y_0 \\ \hline & s_7 & s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \end{array}$$

First observe

$$s_0 = \begin{cases} 0 & \text{if } x_0 + y_0 = 0 \\ 1 & \text{if } x_0 + y_0 = 1 \\ 0 & \text{if } x_0 + y_0 = 2. \end{cases}$$

Equivalently, $s_0 = \text{ODD}(x_0,\, y_0,\, 0)$, where ODD is the 3-way odd parity function discussed in Section 1.2. Now we give a Boolean expression for the carry bit. We will use c_i to denote the carry bit produced when adding the ith digits. It follows that

$$c_0 = \begin{cases} 0 & \text{if } x_0 + y_0 = 0 \\ 0 & \text{if } x_0 + y_0 = 1 \\ 1 & \text{if } x_0 + y_0 = 2. \end{cases}$$

Equivalently, $c_0 = \text{MAJ}(x_0,\, y_0,\, 0)$, where MAJ is the 3-way majority function described in Section 1.2. This method extends to computing the ith digit in the sum, and the ith carry bit. In general

$$s_i = \begin{cases} 0 & \text{if } x_i + y_i + c_{i-1} = 0 \text{ or } 2 \\ 1 & \text{if } x_i + y_i + c_{i-1} = 1 \text{ or } 3. \end{cases} \qquad \text{and} \qquad c_i = \begin{cases} 0 & \text{if } x_i + y_i + c_{i-1} = 0 \text{ or } 1 \\ 1 & \text{if } x_i + y_i + c_{i-1} = 2 \text{ or } 3. \end{cases}$$

Equivalently, $s_i = \text{ODD}(x_i,\, y_i,\, c_{i-1})$ and $c_i = \text{MAJ}(x_i,\, y_i,\, c_{i-1})$. Note that c_{i-1} must be computed before z_i or c_i. We see that the MAJ and ODD circuits play an important role in binary addition.

To build an *n-bit ripple-carry adder*, it is convenient to first define a *1-bit adder* that takes 3 inputs, say x, y, and z, and produces 2 outputs, say S (sum) and C (carry bit). We design the circuit so that $S =$ ODD(x, y, z) and $C =$MAJ(x, y, z). This circuit is described in Figure 12. Now, by chaining together 4 of these 1-bit adders, we can build an 4-bit ripple-carry adder, as in Figure 13. In general, an n-bit adder has $2n$ inputs, n corresponding to the first summand x, and n corresponding to the second summand y. It produces n outputs, corresponding to the sum $s = x + y$. To avoid overflow, we could design an n-bit adder that adds two n-bits integers and return an $(n + 1)$-bit integer. Instead, we truncate the result to n-bits, e.g., 1111 + 0011 = 0010.

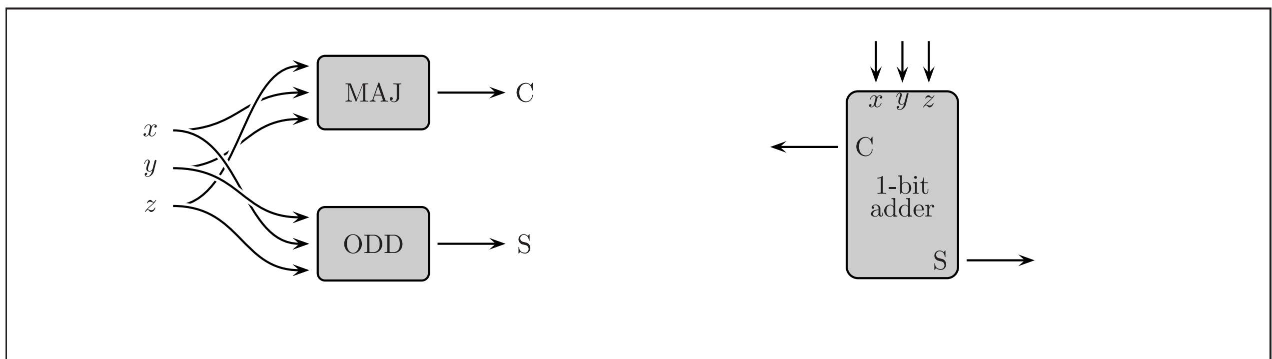


Figure 12: A 1-bit adder and its schematic symbol.

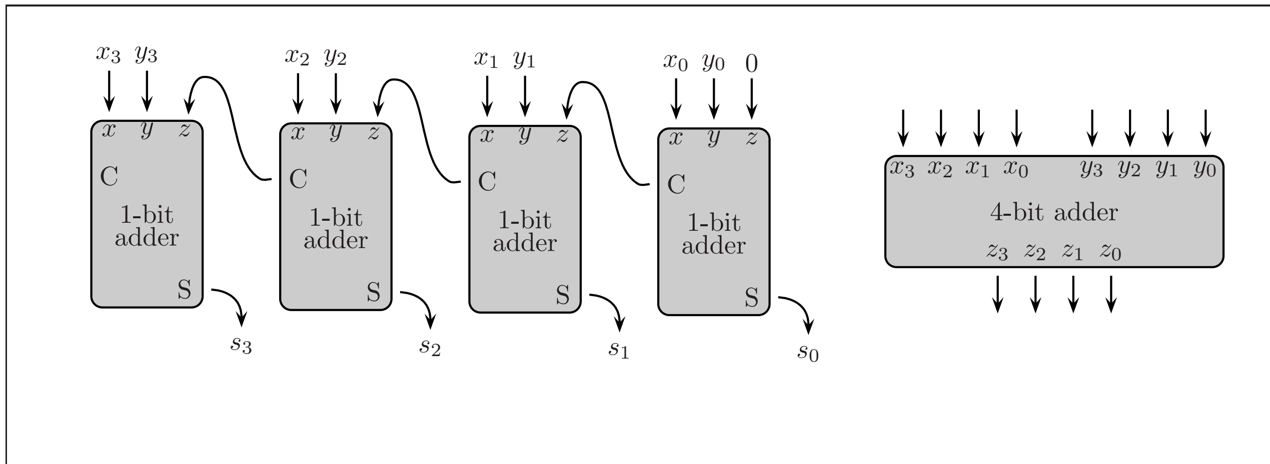


Figure 13: A 4-bit adder and its schematic symbol.

4.7 Subtracter

In the previous section we saw how to build a circuit to perform binary addition. In principle, we could use the exact same approach and build a binary subtracter circuit. In practice, it is simpler to directly reconfigure the adder circuit to perform subtraction.

To subtract y from x, we will perform the addition $x + (-y)$. Recall that with 2's complement integers, to convert from y to $-y$ involves flipping all of the bits, and then adding one. This is particularly easy to implement in hardware. To flip the bits, we use a NOT gate. Adding 1 to a number seems like a bigger challenge. This is where we use the *carry-in* input to the least significant bit. For addition, we always set the carry-in of the least significant bit to be 0, but by changing it to 1, we effectively add 1 to the entire result. Figure 14 gives the details.

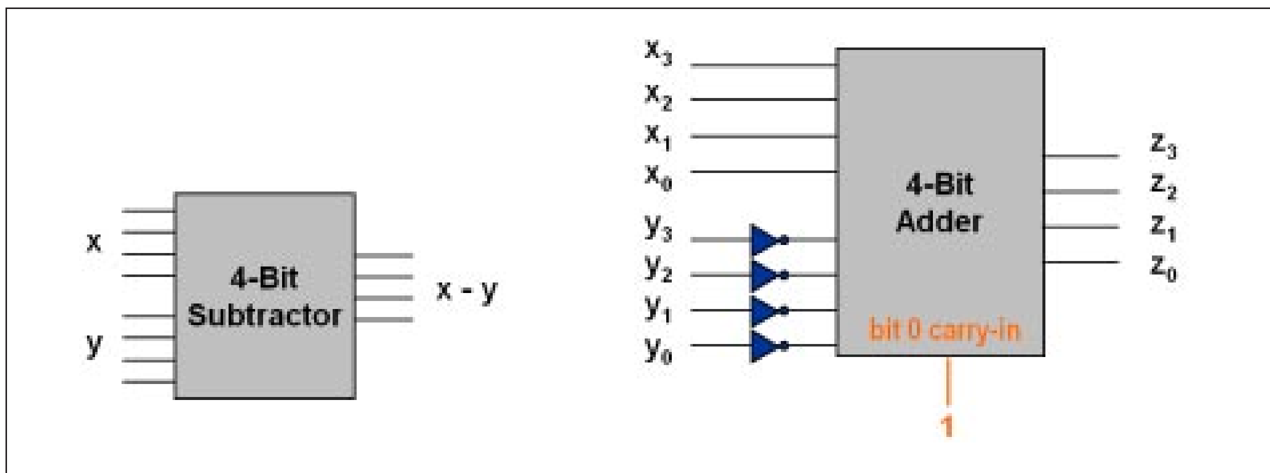


Figure 14: The interface and implementation of a 4-bit subtracter.

5 Arithmetic Logic Unit

The *arithmetic logic unit* (ALU) is the workhorse circuit of a computer. It is responsible for performing the TOY arithmetic operations (addition and subtraction) and the TOY logic operations (bitwise AND, bitwise XOR, shift left, shift right).

The ALU consists of several control inputs and separate circuit components for addition, bitwise AND, bitwise XOR, and shifting. It performs all 4 computations in parallel, and then selects the appropriate output depending on the control inputs.

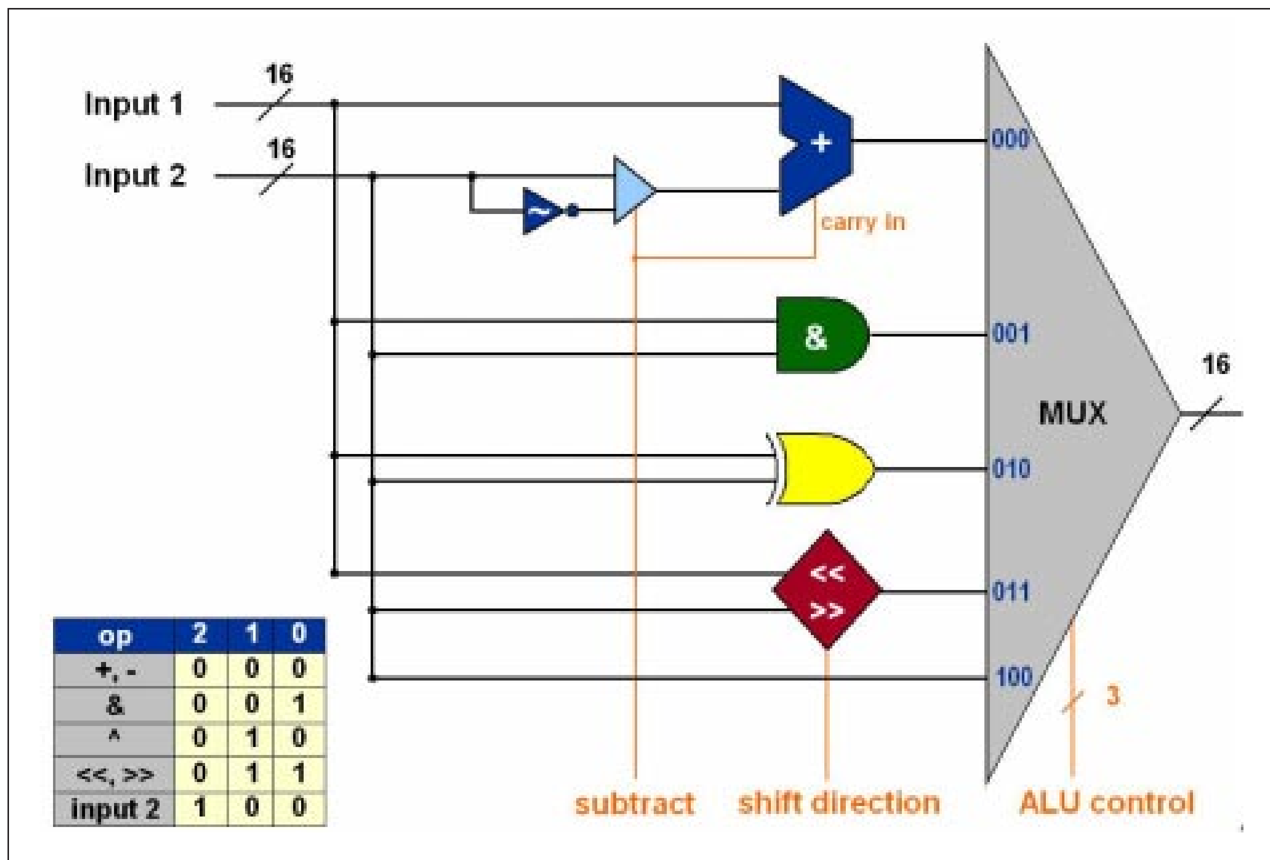


Figure 15: The implementation of the TOY ALU.

6 Sequential Circuits (memory)

Combinational circuits are only capable of performing a computation once, because the values of the gates and wires do not change over time. Thus, if we want to perform ten million additions, we would have to build ten million individual combinational adder circuits. By allowing feedback, the values in a sequential circuit may change over time. This is the essential component needed to make circuits that can "write" and "read" information to and from "memory." This enables sequential circuits to reuse circuit elements, thereby greatly reducing the hardware requirements for building a computer.

Timing. Until now, we have assumed that gates process their input and produce output instantaneously. In fact, it takes a small amount of time for current to transmit through a wire. This *wire timing delay* depends on the length and material properties of the wire. It also takes time for gates to transform their input into output. This *gate timing delay* depends on the configuration, wiring, and number of controlled switches in the gate design. As you would expect, a 3-switch OR gate has a higher gate timing delay than a 1-switch NOT gate.

Understanding the output of a sequential circuit is more difficult than with combinational circuits. With combinational circuits, the input values uniquely determine a value for every gate and wire in the circuit. With sequential circuits, the gate and wire values may change over time. Here's a somewhat simplified view of how a sequential circuit operates. At time t, some of the wires are on, and some are off. The state of the circuit at time $t + dt$ is determined by computing the output of every gate in parallel, using the circuit state at time t. This process is repeated. The state of the circuit at time $t + 2dt$ is determined by computing the output of every gate, but now using the circuit state at time $t + dt$. This process is repeated for each time quantum. For simplicity, we have assumed that the gate timing delay of every gate is exactly dt, and that the wire timing delay is negligible. In reality, each gate has its own timing delay, and wire transmission time is relatively small, but not zero. Subtle differences in timing delays can create bizarre behavior in poorly designed circuits. Consider the circuit in Figure 16. What is the value of the output wire over time? It depends on the timing delays of the gates. The output of the NOT gate constantly changes its value from 1 to 0. As a result, it is not possible to accurately determine the output of the circuit. This is known as a *race condition* and is the bane of circuit design. We will only build circuits that do not directly rely on the timing delays; we only make the reasonable assumption that the timing delays are small, relative to the clock rate.

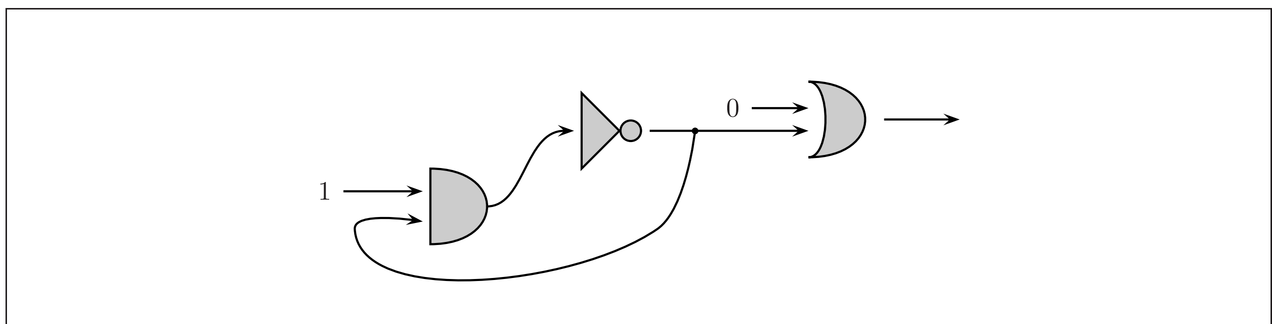


Figure 16: A race condition: the output of the NOT gate oscillates between 0 and 1, so the output of the OR gate depends on the gate timing delays.

6.1 Timing Diagrams

Timing diagrams are important for understanding the output of sequential circuits. For combinational circuits, a truth table characterizes the behavior of the circuit for all possible input combinations. For sequential circuits, it is not possible to enumerate all possible input combinations, since the input is really a stream of inputs that change over time. We use a *timing diagram* to chart the state of the circuit over time for *one particular* input stream. We plot time on the x-axis and the on/off status of the wire on the y-axis.

6.2 Clock

To build computers, we need to be able to synchronize and coordinate various computations. Computers use an external *clock* to accomplish this. The clock is an abstraction that transmits on and off pulses at regular intervals. The pulse lasts for a relatively small amount of time. See Figure 17 for a timing diagram of a typical clock. The clock in a 500MHz Pentium pulses 500 million times a second. The clock speed determines the rate at which instructions are processed, and greatly influences the overall speed of the computer.

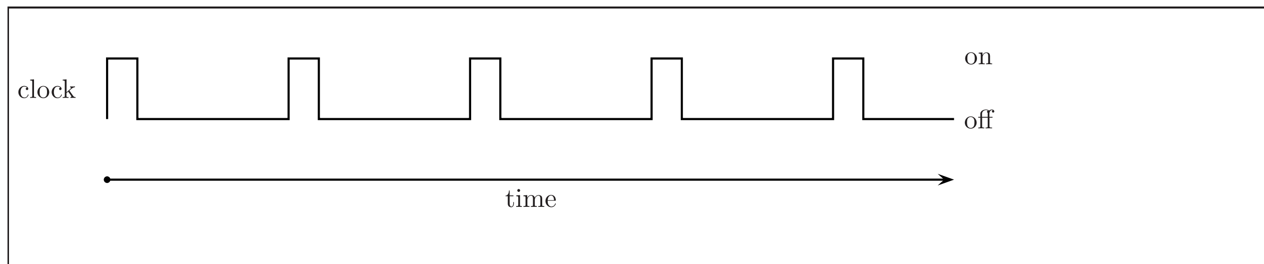


Figure 17: The timing diagram of a clock.

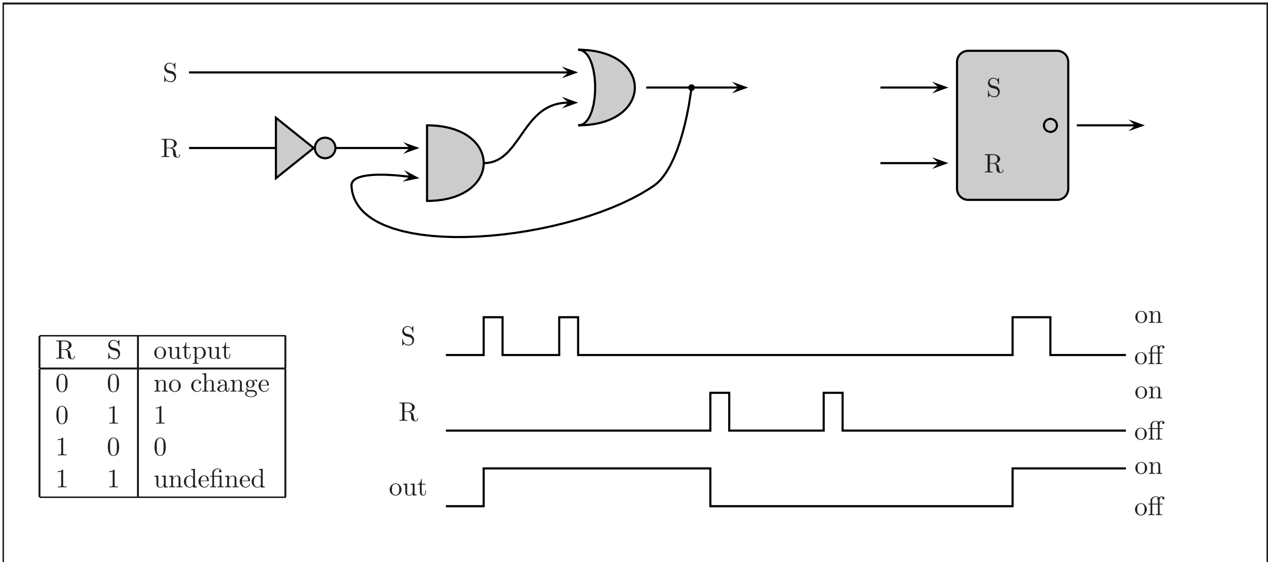
Circuits are designed so that new computations can only be initiated when the clock is on. The computations can proceed while the clock is off, but it is often desirable for it to finish before the next clock pulse. Building circuit elements that will begin and finish execution within a single clock tick depends directly on the speed at which the underlying switches operate. The invention of a faster switch would enables computers to use a faster clock. Different computer architectures use different instruction sets. RISC (reduced instruction set computers) machines can have faster clocks, but this comes at the expense of perfoming a more limited range of instructions per clock cycle.

6.3 Flip-flops

A *flip-flop* is a simple sequential circuit that is capable of "storing a bit." To accomplish this, the flip-flop creates a feedback loop which transmits current (or no current) across the loop. When the feedback loop has current, we interpret it as storing the bit 1; when there is no voltage we interpret it as storing the bit 0. Flip-flops are used to build registers and memory. We consider several types of flip-flops below.

6.4 SR flip-flop

The *SR flip-flop* is the simplest type of flip-flop; the corresponding circuit and schematic symbol are shown in Figure 18. It has two inputs, labeled S (set) and R (reset). It has one output which represents the value of the stored bit. It works as follows. If S and R are both 0 (neither line is selected), then the value propagated around the feedback loop is unchanged. If the bit was 1, then it remains 1; if it was 0, then it remains 0. If S is selected (set to 1), then the value 1 will be transmitted around the feedback loop. If R is selected (set to 1), then the value 0 will be transmitted around the feedback loop. The output is undefined if both S and R are selected at the same time – we will see a slightly more complicated flip-flop that overcomes this potential pitfall.



R	S	output
0	0	no change
0	1	1
1	0	0
1	1	undefined

Figure 18: An SR flip-flop, its schematic symbol, and an sample timing diagram. The output can only change just when S or R is turned on. If S is turned on, the output becomes 1; if R is turned on the output becomes 0.

6.5 Clocked SR flip-flop

A *clocked SR flip-flop* is the same as an SR flip-flop except that all actions are synchronized with the clock. It has three inputs: S (set), and R (reset), and Cl (clock). If the clock is on and S is selected, then the flip-flop will store a 1. If the clock is on and R is selected, then the flip-flop will store a 0. If the clock is off, then the value currently stored in the flip-flop will remain unchanged. A clocked SR flip-flop can be implemented by ANDing the S and R inputs with the clock as in Figure 19.

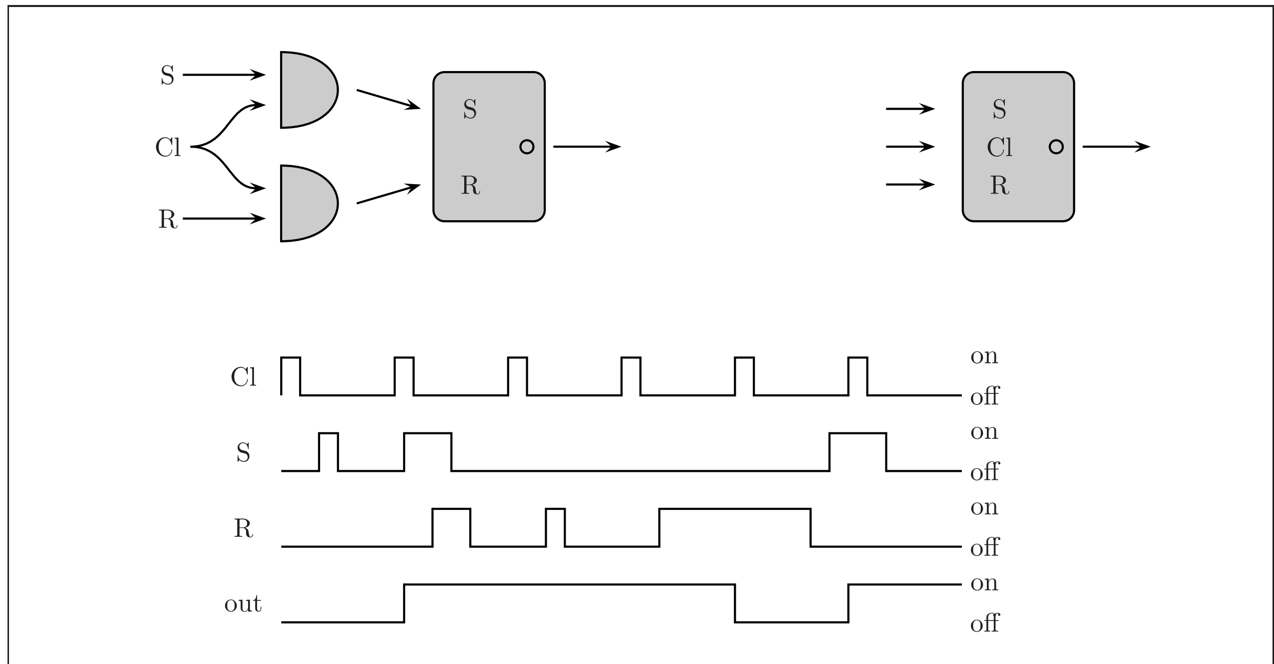


Figure 19: A clocked SR flip-flop, its schematic symbol, and a sample timing diagram. The output can only change when the clock is on.

6.6 Clocked D flip-flop

A *clocked D flip-flop* is useful for directly copying an input bit value into memory. A clocked D flip-flop has two inputs: D (data) and Cl (clock). If the clock is on, then the bit value on D is copied and stored in the flip-flop: if D is 0, then a 0 is stored, if D is 1, then a 1 is stored. If the clock is off, then the value currently stored in the flip-flop remains unchanged. A D flip-flop can be implemented from a clocked SR flip-flop and one NOT gate, as illustrated in Figure 20. The D input is wired directly to the S input of a clocked SR flip-flop, and indirectly through a NOT gate to the R input the SR flip-flop. The clock of the D input is wired directly to the clock input of the SR flip-flop. When D is on, S is on and R is off. When D is off, S is off and R is on. Thus, exactly one of the inputs to the clocked SR flip-flop is always on. If the clock is on, then this effectively copies the bit value of the D wire into the flip-flop "memory." If the clock is off, the memory remains unchanged.

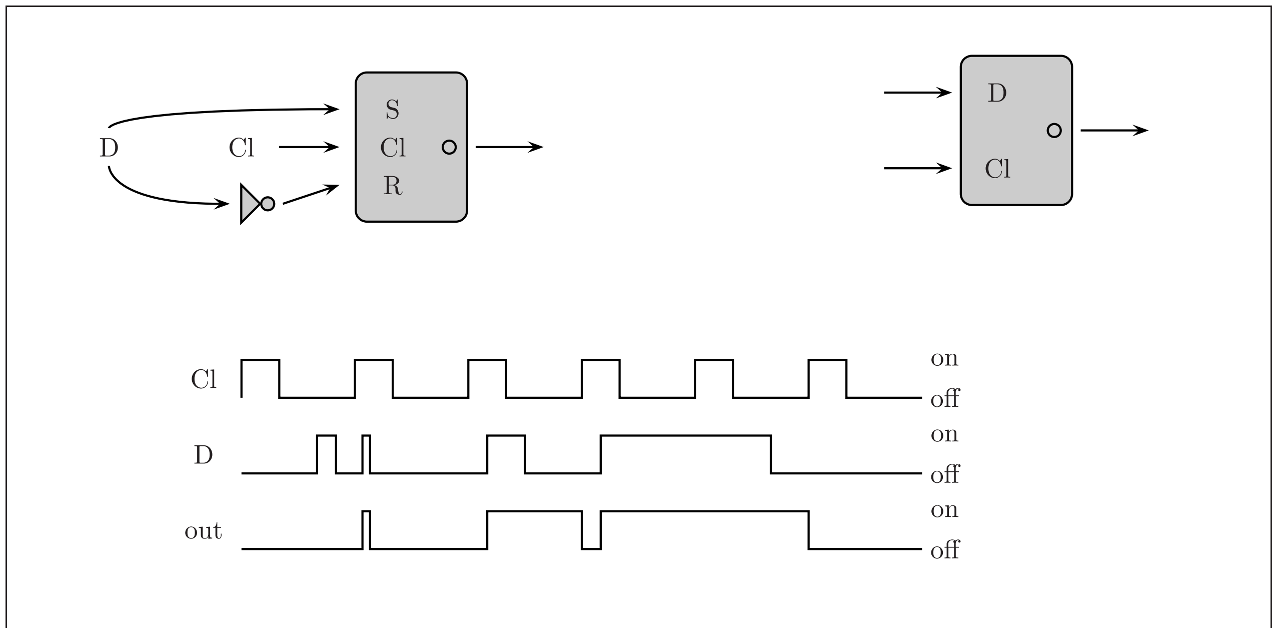


Figure 20: A D flip-flop, its schematic symbol, and a sample timing diagram. The output only changes when the clock is on: the output is the same as D during this period.

6.7 Register

A *n-bit register* is a repository for storing n bits. It has n input values $x_0, \ldots, x_{n-1}$, and n output values $y_-, \ldots, y_{n-1}$. It also has a write input that must be set in order for any data to be written to the register. The values $y_0, \ldots, y_{n-1}$ are stored in the n flip-flops and are always available on the output wires. These values do not change unless the write input is on. If the write input is on, then the values $x_0, \ldots, x_{n-1}$ are written into the n flip-flops. Think of x as being the output of some calculation that we wish to store away in y for later reuse. We implement a register in Figure 21 as a sequence of clocked D flip-flops.

An 8-bit register corresponds directly with the TOY program counter. In the next section we will see how to combine 256 16-bit registers to implement the TOY main memory.

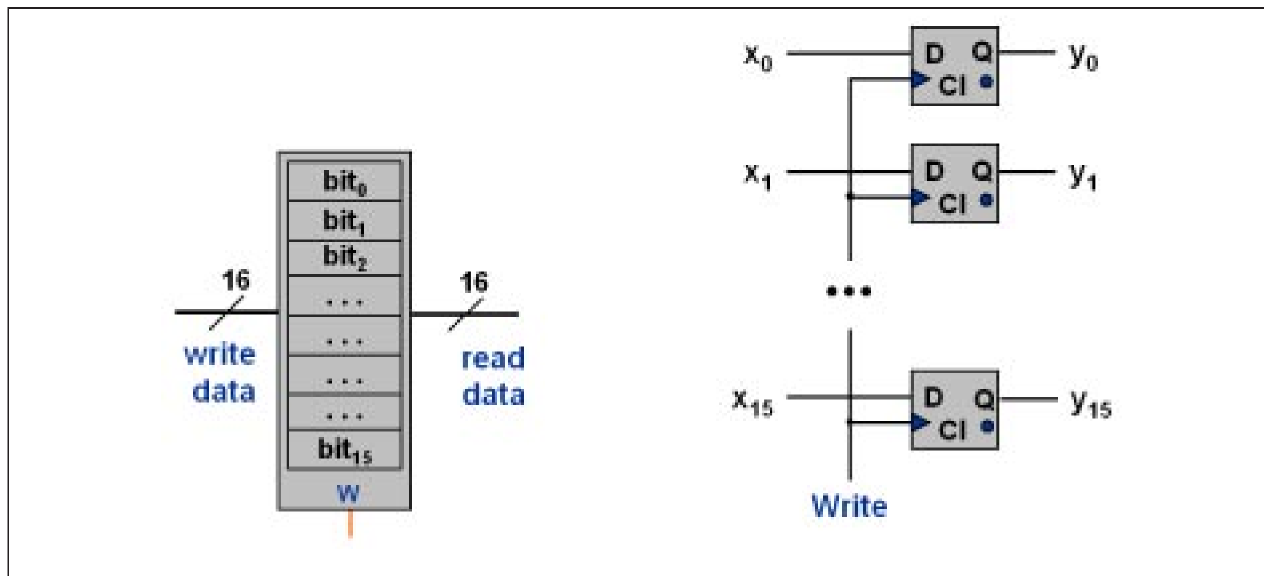


Figure 21: The interface and implementation of a 16-bit register.

6.8 Register file

Modern computers contain millions or billions of bits of memory. A *register file* provides a systematic method to organize and manipulate them. A $k \times n$ *register file* is a bank of n registers, each of which stores k bits. It contains k data inputs, k data outputs, $\log_2 n$ address inputs, a clock input, and a write input. At any point in time, exactly one of the n registers can be accessed: the $\log_2 n$ address inputs specify which one. The contents of this addressed register appear on the k output wires. If the clock and write inputs are on, then the contents of the k input wires will be immediately copied to the addressed register.

We implement a $k \times n$ register file as a sequence of n k-bit registers along with some combinational circuitry to access the registers. A k-wide n-to-1 multiplexer is ideally suited for selecting which register to select as the output. For writing, we used a $\log_2 n$-bit decoder, and wire its n outputs through an AND gate to the write input of the corresponding register. Since the contents of a register can only change if the write input is on, this ensures that only the appropriate register is overwritten. Figure 22 gives the details for a 256×16 register file.

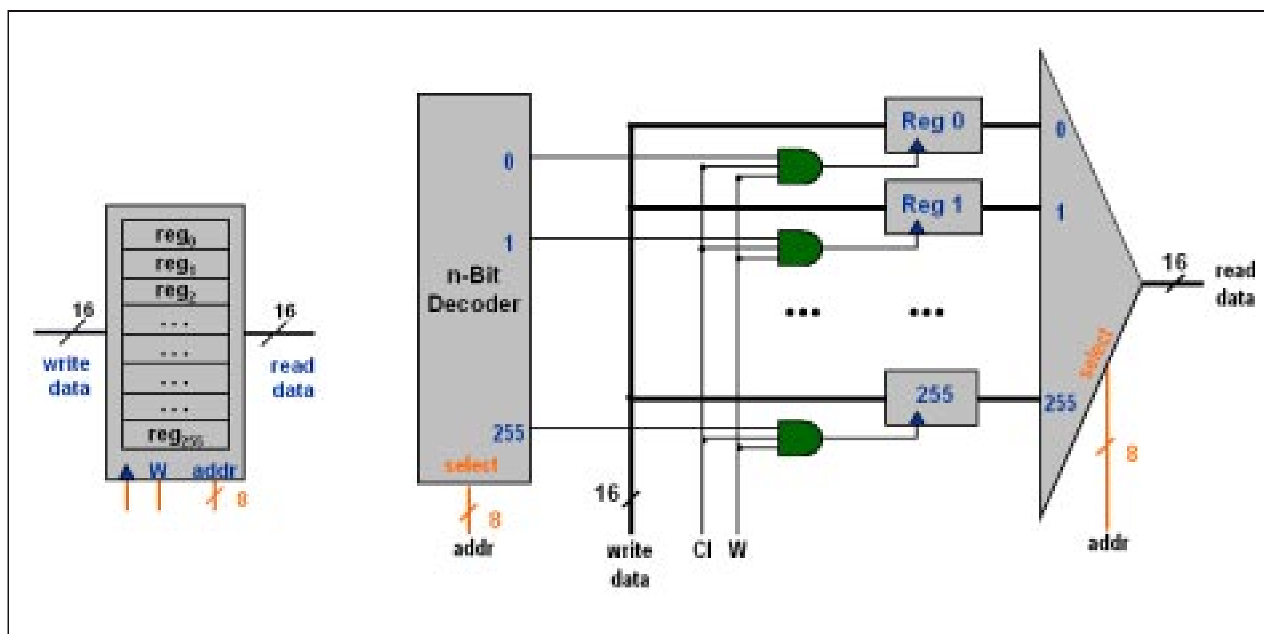


Figure 22: The interface and implementation of a 256 ×16 register file.

6.9 1-bit counter

A *1-bit counter* or *toggle* is a circuit that takes one clock input and has one output such that the output changes (from 0 to 1, or from 1 to 0) each time the input goes from on to off. We will use this circuit to choreograph the fetch-execute cycle of the TOY machine. When the counter's output is 0, the machine will be fetching the next instruction; when it's output is 1, the machine will be executing the current instruction.

We build a 1-bit counter by wiring together two D flip-flops through a NOT gate. Figure 23 describes the interface, implementation, and timing diagram of a *toggle*.

The toggle's output only changes when its clock input goes from on to off. This property enables refined control over when actions will occur. For example, after an instruction like `R[1] <- R[1] + R[2]`, we want to be sure that we don't overwrite the contents of register 1 until the end of the execute cycle; otherwise this updated value could get used in the addition instead of the original value. To ensure that this feedback doesn't occur, we will only allow overwriting a register to occur when the clock and the toggle are both 1, i.e., at the very end of the execute phase.

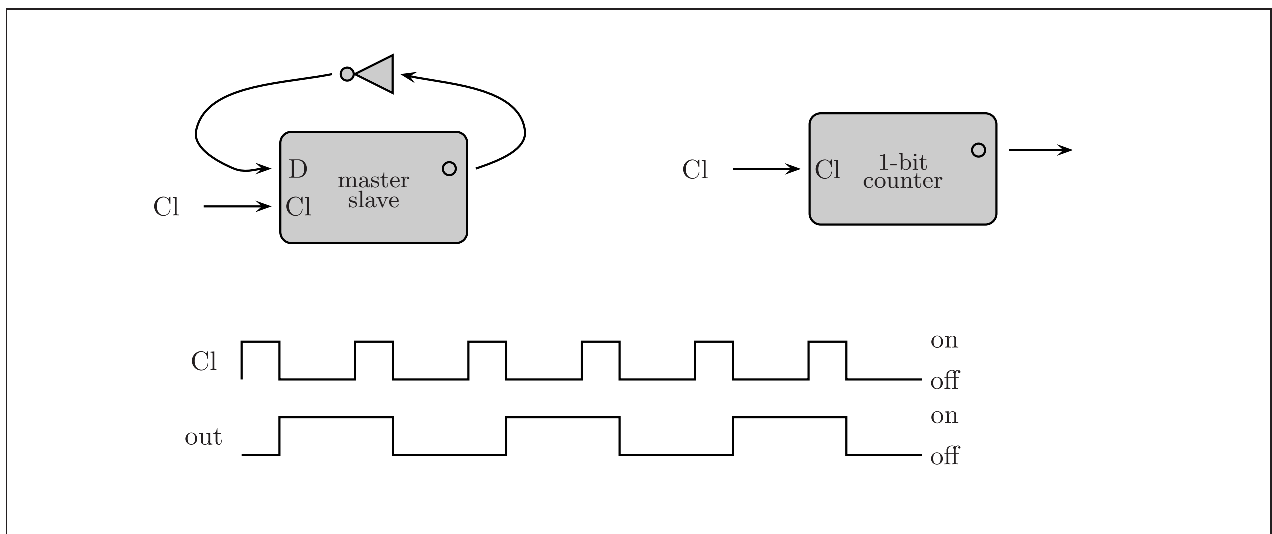


Figure 23: A 1-bit counter, its schematic symbol, and a sample timing diagram. The output changes when the input goes from on to off.

6.10 Counter (optional)

A *n-bit counter* has one input and n outputs. The counter counts the number of times that the input goes from on to off. The n outputs store this quantity as a binary number. Although the counter is a useful circuit in many contexts, we do not use it in the design of the TOY machine.

We implement an n-bit counter by chaining together n 1-bit counters in series, as in Figure 24.[3] The output c_0 is exactly the output of a single 1-bit counter. It goes from 0 to 1 just after the 1st, 3rd, 5th, and 7th clock pulses, i.e., every other clock pulse. Output c_0 is the input to the next 1-bit counter. Thus, the output c_1 changes when c_0 goes from 1 to 0. Hence, c_1 changes from 0 to 1 just after the 3rd, 7th, 11th, 15th clock pulses, i.e., every 4th clock pulse. Similarly, output c_2 changes from 0 to 1 every 8th clock pulse, and in general, c_i changes from 0 to 1 every 2^{i+1} clock pulses.

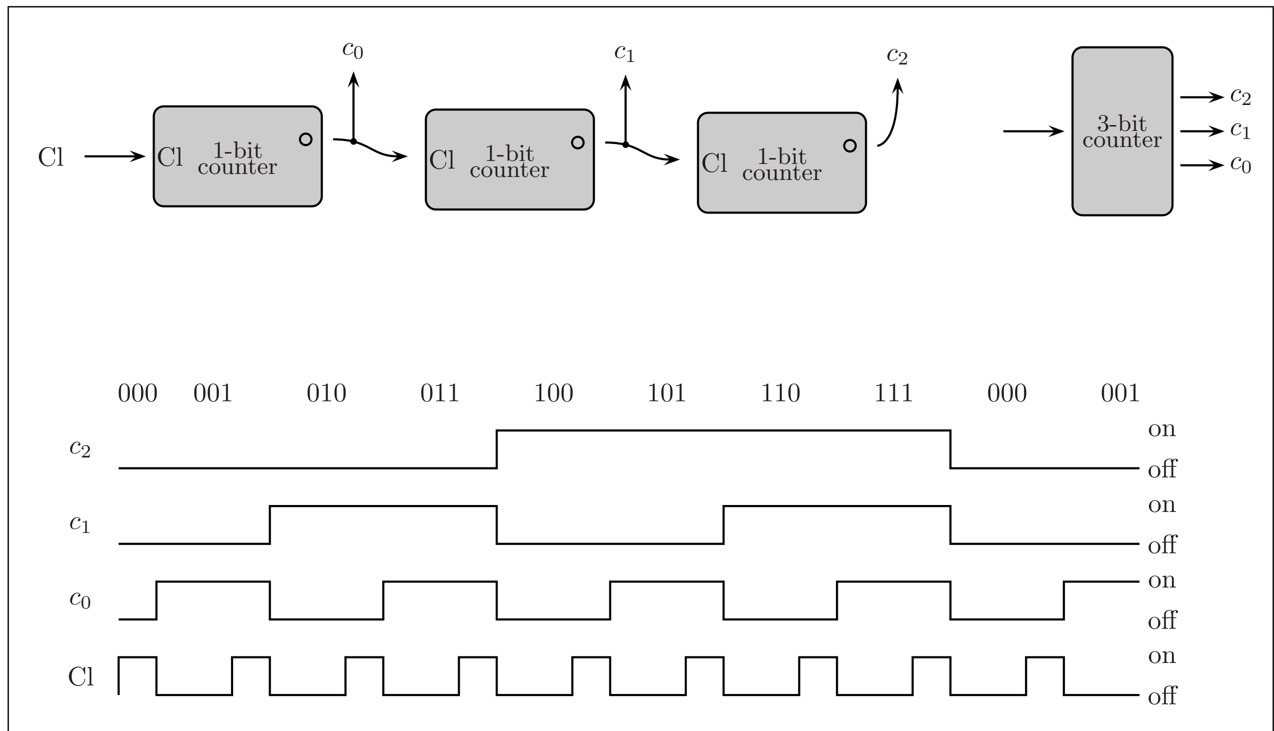


Figure 24: A 3-bit counter, its schematic symbol, and a sample timing diagram.

[3] Although this is the simplest way to build a counter, it is not recommended in practice. Because of timing delays, the n outputs of the circuit are not determined simultaneously; bit 0 is known first, then bit 1, then bit 2, and so on.

7 Building a TOY Machine

We have described all of the components need to build the TOY computer: ALU, clock, counter, registers, and register files. We will now describe how to assemble these pieces and produce a blueprint for the TOY machine. There are 3 remaining steps.

1. Determine datapath requirements. We need to ensure that all of the data can be routed to the appropriate place.

2. Establish clocking methodology. The TOY machine has two phases: fetch and increment. Both phases may to access the same hardware resources. For example, the fetch phase needs to read from main memory to grab the next instruction to be executed. Executing an TOY load instruction also requires access to main memory. We need to choreograph the fetch-execute cycle so that they do not interfere with each other.

3. Analyze how to implement each instruction, and determine settings of control signals.

Unfortunately, we do not have any notes on this material. Please consult the lecture notes.