

# 7.9 Cryptology



Enigma machine

"Cryptography used to be an obscure science, of little relevance to everyday life. Historically, it always had a special role in military and diplomatic communications. But in the Information Age, cryptography is about political power, and in particular, about the power relationship between a government and its people. It is about the right to privacy, freedom of speech, freedom of political association, freedom of the press, freedom from unreasonable search and seizure, freedom to be left alone." - Phil Zimmermann

- Turing machines. Newtonian mechanics.
- Computability. Heisenberg uncertainty principle.
- NP-completeness. Speed of light.

This lecture.

- Exploit hard problems.
- Apply theory to cryptography.
- RSA cryptosystem.

"It is insufficient to protect ourselves with laws. We need to protect ourselves with mathematics."  
-- Bruce Schneier

## Cryptology

- Cryptology: science of secret communication.
- Cryptography: science of creating secret codes.
- Cryptanalysis: science of code breaking.



Goal: information security in presence of malicious adversaries.

- Confidentiality: keep communication private.
- Integrity: detect unauthorized alteration to communication.
- Authentication: confirm identity of sender.
- Authorization: establish level of access for trusted parties.
- Non-repudiation: prove that communication was received.

## A Better Approach

Security by obscurity.

- Rely on proprietary, ad hoc cryptographic schemes.
- Eventually reverse-engineered and cracked.
- Ex: CSS for DVD encryption, RIAA digital watermarking, GSM cell phones, Windows XP product activation, Adobe eBooks, Diebold AccuVote-TS machines, . . . .

A better approach.

- Leverage theory of hard problems.
- Show that breaking security system is equivalent to solving some of the world's greatest unsolved problems!

Kerckhoffs' principle.

"The system must not require secrecy and can be stolen by the enemy without causing trouble." ☒☒☒

## Analog Cryptography

Task	Description
Protect information	Code book, lock + key
Identification	Driver's license, fingerprint, DNA
Contract	Handwritten signature, notary
Money transfer	Coin, bill, check, credit card
Public auction	Sealed envelope
Poker	Cards with concealed backs
Public election	Anonymous ballot
Public lottery	Dice, coins
Anonymous communication	Pseudonym, ransom note



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## Digital Cryptography

### Our goal.

- Implement all tasks digitally and securely.
- Implement additional tasks that can't be done with physics!

### Fundamental questions.

- Is any of this possible?
- How?

### Today.

- Give flavor of modern (digital) cryptography.
- Implement one of these tasks.
- Sketch a few technical details.

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## Digital Cryptography Axioms

### Axiom 1. Players can toss coins.

- Crypto impossible without randomness.

### Axiom 2. Players are computationally limited (poly-time).

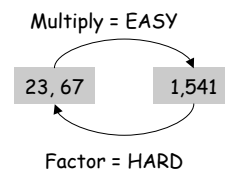
### Axiom 3. Factoring is hard computationally.

- Not polynomial-time.
- "1-way trapdoor function."

### Fact. Primality testing is easy computationally.

### Theorem. Digital cryptography exists.

### Corollary. Can do all tasks on previous slide **digitally**.



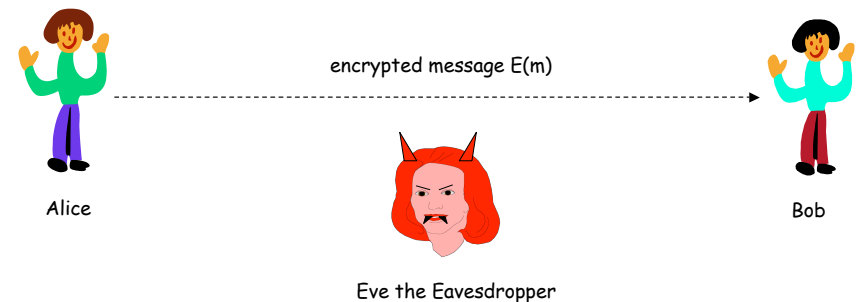
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## Encryption

### Encryption.

- Most basic problem in cryptography.
- Alice sends Bob an encrypted message  $E(m)$ .
- Easy for Bob to recover original message  $m$ .
- Hard for Eve to learn anything about  $m$ .

credit card number

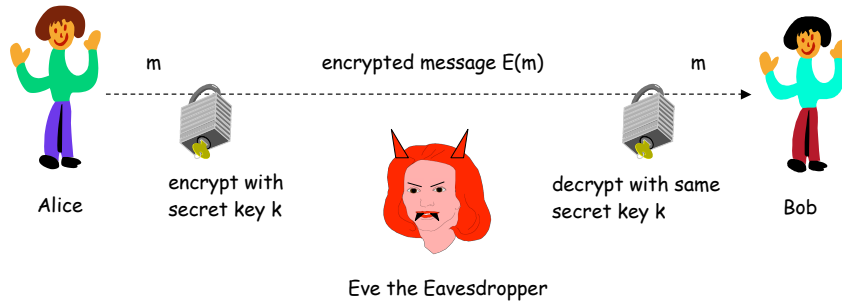


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## Private Key Encryption

Alice sends Bob a message  $m$ .

- Assume message  $m$  encoded in binary.
- Alice and Bob share secret key  $k$ .



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## Private Key Encryption: One Time Pad

Key distribution.

- Alice and Bob share  $n$ -bit secret key  $k$ .

0	1	1	1	0	0
$n = 6$					

$k$

Alice wants to send  $n$ -bit message  $m$  to Bob.

- Alice computes and sends  $E(m) = m \hat{ } k$ .
- ↑  
bitwise XOR

0	1	0	1	1	0
0	0	1	0	1	0

$m$   
 $E(m)$

Bob receives ciphertext  $c = E(m)$ .

- Bob computes  $D(c) = c \hat{ } k$ .

0	0	1	0	1	0
0	1	0	1	1	0

$c$   
 $D(c)$

Why does it work?  $D(E(m)) = D(m \hat{ } k) = (m \hat{ } k) \hat{ } k = m$

Why is it secure? If  $k$  is uniformly random, so is  $m \hat{ } k$ .

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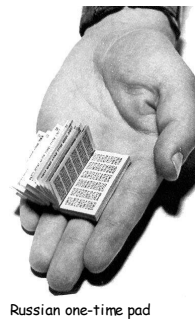
## Private Key Encryption

Advantages.

- Provably secure if key is random.
- Simple to implement.

Disadvantages.

- Not easy to generate uniformly random keys.
- Need new key for each message.
- Signature? ← Rosenbergs sent to electric chair because Russian spy reused a one-time pad
- Non-repudiation?
- Key distribution? ← deal-breaker for e-commerce since Alice and Bob want to communicate even if they've never met



Other private key encryption schemes.

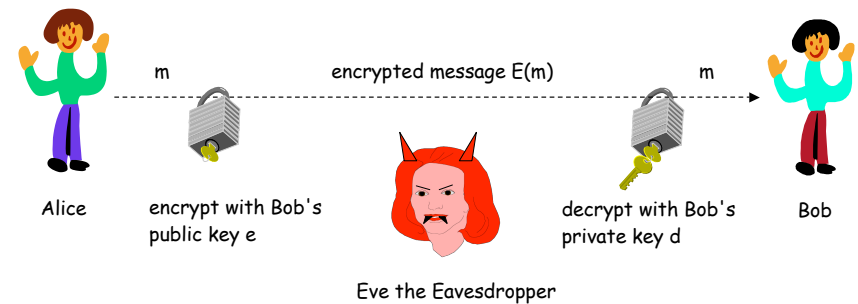
- Data Encryption Standard (DES).
- Advanced Encryption Standard (AES, Rijndael algorithm).
- Blowfish.

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## Public Key Encryption

Alice sends Bob a message  $m$ .

- Bob has **public** key  $e$  and **private** key  $d$ .
- ↑ locks                      ↑ unlocks



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## Public Key Encryption

### Key distribution.

- Bob has **public** key = published in digital phonebook.
- Bob has **private** key = known only by Bob.

VeriSign  
↓

### Alice wants to transmit N-bit private message $m$ to Bob.

- Alice encrypts message using Bob's public key:  $E(m)$ .

### Bob receives ciphertext $c = E(m)$ from Alice.

- Bob decrypts message using his private key:  $D(c)$ .

Under what situations does it work?  $D(E(m)) = m$ . ← absolute and obvious requirement

### What are necessary conditions for security?

- Can encrypt message efficiently with public key.
- Can decrypt message efficiently with private key.
- Can **not** decrypt message efficiently with public key alone.

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## RSA Public Key Cryptosystem: In the Real World

### RSA cryptosystem (1978).



Rivest Shamir Adleman

### Operating systems. Sun, Microsoft, Apple, Novell.

**Hardware.** Cell phones, ATM machines, wireless Ethernet cards, Mondex smart cards, Palm Pilots, Palladium.

**Secure Internet communication.** Browsers, S/MIME, SSL, S/WAN, PGP, Microsoft Outlook, etc.

Alice browses to <https://whiteboard.cs.princeton.edu>  
 Alice's browser gets Bob's public key.  
 Alice sends programming assignment.  
 Bob's web server decrypts assignment.

Alice submits programming assignment to Bob via secure website

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## RSA Public-Key Cryptosystem: Key Generation

### RSA key generation.

- Select two large prime numbers  $p$  and  $q$  at random.
- Compute  $N = pq$ .

$$p = 11, q = 29$$

$$N = 11 \times 29 = 319$$

**Number theory fact.** If  $p$  and  $q$  are prime, there exist efficiently computable integers  $e$  and  $d$  such that for all messages  $m$ :  $(m^e)^d \equiv m \pmod{N}$ .

$$(m^3)^{187} \equiv m \pmod{319}$$

$a \equiv b \pmod{N}$  means  $(a \% N) == (b \% N)$

Bob's public key:  $(e, N)$

$$(3, 319)$$

Bob private key:  $(d, N)$

$$(187, 319)$$

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## RSA Public-Key Cryptosystem: Encryption and Decryption

### Alice wants to transmit $n$ -bit private message $m$ to Bob.

- Alice obtains Bob's public key  $(e, N)$  from Internet.
- Alice computes  $E(m) = m^e \pmod{N}$ .

$$E(m) = 100^3 \pmod{319} = 254$$

### Bob receives ciphertext $c$ from Alice.

- Bob uses his secret key  $(d, N)$ .
- Bob computes  $D(c) = c^d \pmod{N}$ .

$$m = 100$$

$$D(c) = 254^{187} \pmod{319} = 100$$

### Why does it work?

- Need to check that  $D(E(m)) = m$ .

$$\begin{aligned} D(E(m)) &= D(m^e) \pmod{N} \\ &= (m^e)^d \pmod{N} \\ &= m \pmod{N} \end{aligned}$$

↑  
previous fact

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## Modular Exponentiation: Brute Force

Modular exponentiation:  $c = a^b \pmod{N}$ .

```
200317 (mod 3713) = 134454746427671370568340195448570911966902998629125654163 (mod 3713)
                  = 232
```

Brute force: multiply a by itself, b times.

Analysis of brute force.

- Suppose a, b, and N are n-bit integers.
- Problem 1: number of multiplications proportional to  $2^n$ .
- Problem 2: number of digits of intermediate value can be  $2^n$ .
- Exponential time and memory!

↑  
bad news since n must be big for RSA to be secure

↑  
128TB memory if N = 50

## Modular Exponentiation: Repeated Squaring

Idea 1: can mod out by N after each multiplication.

- Intermediate numbers stay small.

Idea 2: repeated squaring.

```
200317 (mod 3713)
= 20031 × 200316 (mod 3713)
= 2003 × 3157 (mod 3713)
= 6,323,471 (mod 3713)
= 232 (mod 3713)
```

$17_{10} = 10001_2$

Term	Compute	mod 3713
2003 <sup>1</sup>	2003	2003
2003 <sup>2</sup>	2003 <sup>2</sup>	1969
2003 <sup>4</sup>	1969 <sup>2</sup>	589
2003 <sup>8</sup>	589 <sup>2</sup>	1612
2003 <sup>16</sup>	1612 <sup>2</sup>	3157

Repeated squaring

Analysis of modular exponentiation.

- At most  $2n$  multiply and mod operations.
- Intermediate numbers at most  $2n$  digits long.

## RSA Details

How large should  $n = pq$  be?

- 2,048 bits for long term security.
- Too small  $\Rightarrow$  easy to break.
- Too large  $\Rightarrow$  time consuming to encrypt/decrypt.

Q. How do I choose a large "random" prime number?

A. Guess-and-check.

Prime Number Theorem. (Hadamard, Vallée Poussin, 1896).

- Number of primes between 2 and  $N \approx N / \ln N$ .
- Primes are plentiful:  $10^{151}$  with  $\leq 512$  bits.
- Will never run out, and no two people will pick same ones.

Theorem. (Agarwal-Kayal-Saxena, 2002)

- PRIME: Given n-bit integer N, is N prime?
- PRIME is in P.

## RSA in Java

Key generation using: `java.math.BigInteger`, `java.security.SecureRandom`.

```
SecureRandom random = new SecureRandom();

BigInteger ONE = new BigInteger("1"); // random n/2-bit prime
BigInteger p = BigInteger.probablePrime(n/2, random);
BigInteger q = BigInteger.probablePrime(n/2, random);
BigInteger phi = (p.subtract(ONE)).multiply(q.subtract(ONE));

BigInteger N = p.multiply(q); // modulus
BigInteger e = new BigInteger("65537"); // public key
BigInteger d = e.modInverse(phi); // private key
// (ed = 1 mod phi)
```

RSA function.

```
BigInteger rsa(BigInteger a, BigInteger b, BigInteger N) {
    return a.modPow(b, N);
} // built-in modular exponentiation (repeated squaring)
```

## Cryptanalysis: RSA Attacks

**Factoring.** Factor  $N = pq$ . Use  $p$ ,  $q$ , and  $e$  to compute  $d$ .

**Other means?** Long-standing open research question. No guarantee that RSA is secure even if factoring is hard.

**Semantic security.** If you know Alice will send ATTACK or RETREAT you can encrypt ATTACK and RETREAT using Bob's public key, and check which one Alice sent.

**Timing attack.** Alice gleans information about Bob's private key by measuring time it takes Bob to exponentiate.

**Modulus sharing.**

- Bob:  $(d_1, e_1, N)$ , Ben:  $(d_2, e_2, N)$ .
- Bob can compute  $d_2$  given  $e_2$ ; Ben can compute  $d_1$  given  $e_1$ .

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## RSA Tradeoffs

**Advantages.**

- Solves key distribution problem.
- Extends to digital signatures, etc.

**Disadvantages.**

- Security relies on decryption being "computationally inefficient."
- Not semantically secure.
- Decryption more expensive than private key schemes.

no such reliance with one-time pads



**Practical middle-ground hybrid system.**

- Use AES, a private key encryption system.
- Use RSA to distribute AES keys.

**Theoretical high-ground. (Blum-Goldwasser, 1985)**

- Provably as hard a factoring.
- Semantically secure.

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## Consequences of Cryptography

**Crypto liberates (you = Alice or Bob).**

- Freedom of privacy, speech, press, political association.
- Benefits both ordinary citizens and terrorists.

**Crypto enables e-commerce.** confidentiality, integrity, authentication.

*Encrypting transactions on the Internet is the equivalent of arranging an armored car to deliver credit-card information from someone living in a cardboard box to someone living on a park bench. -- Eugene Spafford*

**Crypto restricts (you = Eve, your computer = Alice or Bob).**

- Ex: Trustworthy Computing, DRM.
- Establishes a secure identity and enable secure transactions.
- Restricts what user can do: play MP3 files, copy DVDs, run software, print documents, forward email.

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